GOVERNING EQUATIONS OF PLASTIC DEFORMATION OF A GRANULAR MEDIUM

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Linear tensor relationships between the stresses and strain rates in an isotropic plastic material contain two independent scalar functions. It is proposed to utilize two independent limit conditions to determine them: One is formulated in stress space (one of the forms of the Coulomb condition), and the other in the strain rate space (dilatancy relationship). Work-hardening is taken into account, the problem of the shear strain of a layer of sand is examined, the proposed governing equations are discussed comparatively. For simplicity, the analysis is restricted to plane motion.

1. Initial representations. A medium comprised of solid, slightly elastic particles (granules) is examined. Upon the application of loads, the medium is deformed so that the strains associated with the relative slippage of the granules play the dominant part. In this case, dry friction forces subiect to the Coulomb law

$$|R| = N \operatorname{tg} \vartheta \tag{1.1}$$

act at the contacts between the granules, where N, R are the normal (compressive) and tangential components of the forces acting on the slippage surface, and ϑ is the true angle of dry friction (between the particles).

Although the velocity field in microscale contains a set of tangential discontinuities because of slippage of the granules, let us introduce a continuous field of mean velocities $u_i\,(x_j,\ t)$. Then the distribution of the relative velocities Δu_i of particles separated by a spacing l in the neighborhood of the macropoint x_j (taking account of rotation of the neighborhood as a rigid whole is not essential to the subsequent analysis) is characterized by the strain rate tensor ε_{ij}, calculated by means of the displacement velocities $u_i\,(x_j,\ t)$ in the customary manner [1]

$$\varepsilon_{ij} = {}^1/_2\,(\partial u_i / \partial x_j + \partial u_j / \partial x_i) \tag{1.2}$$

In fact

$$\Delta u_i = u_i\,(x_j + l_j,\ t) - u_i\,(x_j,\ t) = \sum_j \varepsilon_{ij} l_j \tag{1.3}$$

These relative displacements are due to the effect of the stresses σ_{ij}.

Let us seek the relations $\sigma_{ij} = f\,(\varepsilon_{ij};\ \varphi,\ \Lambda,\ \chi,\ \ldots)$ governing the mechanical behavior of a granular medium during its plastic deformation. Here φ, Λ, χ $\cdots$ are the undetermined parameters of the state of the medium. Seeking the relations between the stress σ_{ij} and strain rate ε_{ij} tensors corresponds to the methods of incremental plasticity theory [1, 2].

The starting point for what follows will be an isotropic linear tensor relation between

σ_{ij} and ε_{ij}, modified for the case of the plane problem

$$\varepsilon_{ij} = 2\lambda^{\cdot}(\sigma_{ij} + p\delta_{ij}) - \zeta^{\cdot}(2H + 2p)\delta_{ij} \tag{1.4}$$

where H is some as yet undetermined quantity with dimension of stress. The relation (1.4) includes [3, 4] two independent scalars $\lambda^{\cdot}$, $\zeta^{\cdot}$ which are characteristic for a plastic material. They can be functions of the state parameters, invariants of the tensor σ_{ij} and other parameters, including hardening [1], if the effect of hardening is essential.

Let us consider the functions $\lambda^{\cdot}$, $\zeta^{\cdot}$ to be different from zero if two limit conditions are satisfied [3, 4]. In contrast to [3, 4] let us formulate just one of the conditions for the stress tensor components; we form the second for the strain rate tensor components [5].

Let us utilize the flow condition (loading) traditional for the theory of plasticity as the first condition

$$\Phi_\sigma(p,\ T;\ \varphi,\ c,\ \ldots) = 0 \tag{1.5}$$

$$p = -{}^1/_2(\sigma_{11} + \sigma_{22}), \qquad T = {}^1/_2\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}$$

by considering the parameters φ, c, ... therein functions of the hardening parameter χ.

The second limit condition is kinematic in nature; let us give it in the form of a relationship such as

$$\Phi_\varepsilon(\varepsilon,\ \gamma;\ \Lambda, \ldots) = 0 \tag{1.6}$$

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22}, \qquad \gamma^{\cdot} = \sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + 4\varepsilon_{12}^2}, \qquad \Lambda = \Lambda(\chi)$$

Condition (1.6) means that the increments in volume and shear are interrelated [6] and this reflects the effect of dilatancy, made apparent experimentally by Reynolds [7] for granular media.

Now, let us consider the stress plane (the pressure p is laid off along the horizontal, and the tangential stress intensity T along the vertical axis). A point on one surface of a family of flow surfaces (1.5) will correspond to each plastic state. Let us also plot the plastic volume strain e along the horizontal axis, and the plastic shear γ along the vertical axis. Then the condition (1.6) connects the projections of the vector of the plastic strain increments $de = \varepsilon dt$ and $d\gamma = \gamma^{\cdot} dt$, i.e., determines the orientation of the vector de_{ij} at the point under consideration relative to the flow surfaces (1.5).

It should here be stressed that the surface to which the vector de_{ij} is orthogonal can be found from the known condition (1.6), i.e., the plastic potential Ψ corresponding to the selected kinematic condition can be found. The corresponding equipotential should not coincide absolutely with the flow surface (1.5). Hence, the kinematic condition (1.6) is very much broader than the demand of normality of the vector de_{ij} to the flow surface, i.e., that particular form of the associative law which is used extensively in the theory of plasticity, and in particular in soil mechanics. The kinematic condition (1.6) corresponds to the general form of the associative law [2].

Upon substitution of (1.4) into the kinematic condition (1.6), the latter is converted into

$$\Phi_\varepsilon(\varepsilon_{ij}(\sigma_{ij})) = \Phi_\varepsilon(\sigma_{ij};\ \lambda^{\cdot},\ \zeta^{\cdot},\ H) = 0 \tag{1.7}$$

which permits expressing $\zeta^{\cdot}$ in terms of $\lambda^{\cdot}$ either directly, or by using the flow condition (1.5). In this latter case, the connection between $\lambda^{\cdot}$ and $\zeta^{\cdot}$ as well as the quantity H are selected from the requirement

$$\Phi_\varepsilon(\sigma_{ij};\ \lambda^\cdot,\ \zeta^\cdot,\ H) \equiv \Phi_\sigma(\sigma_{ij}) = 0$$

In the theory of plasticity developed in application to the strain of metals, the requirement for plastic incompressibility $\varepsilon = 0$ has been proposed, which would be one of the possible kinematic conditions (1.6). It hence follows at once that $\zeta^\cdot \equiv 0$. If $\Phi_\sigma = T - \text{const} = 0$, then it turns out that the vector de_{ij} has just one, nonzero, projection $d\gamma$ and is orthogonal to the flow surface [8].

The factors $\zeta^\cdot$, $\lambda^\cdot$ were introduced in [3, 4] and two limit conditions were sought to determine them, however, in the stress space for both, and this was not successful. It was mentioned in [6] that just the factor $\lambda^\cdot$ turns out to be independent in dilating media and plastic strains were introduced, whose volume increments are proportional to the shear increments.

The use of a dilatancy relation (in the ε_{ij} space) explicitly to express one of the parameters $\lambda^\cdot$, $\zeta^\cdot$ in terms of the other was proposed in [5, 9]. Attention was turned in [10, 11] to the presence of several independent scalar functions in the governing relations for soils, but no condition of general type was advanced to supplement the flow condition (2.4). On the other hand, a particular form of the dilatancy relation (1.6) was introduced [12, 13] but no means has been found to utilize it to decipher the governing relations.

2. Concretization of the limit conditions. Let us use the Coulomb condition as the flow (loading) condition

$$\Phi_\sigma = {}^1/_2 \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}{}^2} + {}^1/_2 (\sigma_{11} + \sigma_{22}) \sin\varphi - c\cos\varphi = 0 \qquad (2.1)$$

where φ is the effective angle of internal friction, and c the cohesion, which are functions of the hardening parameter. The surface (2.1) in pT space is a Mohr-Coulomb line whose slope to the p axis equals $\sin\varphi$.

The results of known experimental researches on the deformation properties of granular media (sand, gravel, etc.) indicate the following. Firstly, there is a domain in the pT stress plane adjacent to the pressure axis p in which the medium behaves elastically (or at least the reversible part of the strain does not exceed the elastic part). This domain is separated from the domain of primarily irreversible strains by the initial flow surface $\Phi_\sigma = 0$. In the majority of the researches it is asserted that this surface (at least, a significant part of it) is a Mohr-Coulomb line (see Sect. 3 below). As regards the nature of these large irreversible strains, the mellow sands are compacted to a certain limit during shear, which corresponds to the critical state at which the medium becomes plastically incompressible. Compact sands, on the other hand, mellow to the critical state during shear. The shear strains apparently grow without limit at the critical state, i.e. disaggregation occurs. As the medium irreversibly compacts (mellows) the flow (loading) surface Φ_σ varies so that the point of the corresponding plastic state is always on the instantaneous loading surface.

Let us utilize the dilatancy constraint [5]

$$\Phi_\varepsilon = (\varepsilon_{11} + \varepsilon_{22}) - \Lambda \sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + 4\varepsilon^2{}_{12}} = 0 \qquad (2.2)$$

as the kinematic limit condition (1.6), where $\Lambda\ (\chi)$ is the effective rate of dilatancy. Let us assume that the hardening parameter χ is an irreversible volume strain or density (such a substitution is possible in the case of a rigidly plastic model). The hardening will therefore be considered isotropic.

If conditions (2.1), (2.2) are not satisfied, then the medium is rigid. The rate of plastic strain ε_{ij} of a granular medium is not zero, i.e., $\lambda^{\cdot} \neq 0$, $\zeta^{\cdot} \neq 0$, if conditions (2.1), (2.2) are satisfied simultaneously. Substitution of the relationship (1.4) into the dilatancy relation (2.2) results in

$$\lambda^{\cdot}\Lambda \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} - \zeta^{\cdot}(\sigma_{11} + \sigma_{22} - 2H) = 0$$

The requirement that the latter agree with the Coulomb limit condition (2.2) results in the dependences

$$\zeta^{\cdot} = -\lambda^{\cdot}\Lambda\sin\varphi, \quad H = c \operatorname{ctg}\varphi \tag{2.3}$$

Hence, the governing equations (1.4), being written for the strain increments de_{ij}, i.e., in the customary form for a plastically hardening model, become

$$de_{ij} = 2(\sigma_{ij} + (1 + \Lambda\sin\varphi)\, p\delta_{ij} + c\Lambda\cos\varphi\delta_{ij})\, d\lambda \tag{2.4}$$

The increment of just one additional scalar function $d\lambda$ figures here. As usual, $d\lambda = 0$ for $\Phi_\sigma < 0$, and also for neutral loading: $d\lambda > 0$ for active loading. The concepts of neutral and active loadings in the domain of the plastic states ($\Phi_\sigma = 0$) will be illustrated by a specific example below.

The effective dependences $\Lambda(\rho / \rho_*)$ and $\varphi(\rho / \rho_*)$ should be chosen from test results. Here ρ is the running value of the density, ρ_* is the density of the reference state. If $\Lambda = 0$ for $\rho = \rho_*$, then ρ_* can be called the critical density. In conformity with the above, we shall call sands mellow at the beginning of this section if $\rho < \rho_*$, $\Lambda < 0$, and compact if $\rho > \rho_*$, $\Lambda > 0$.

Some tests [14] indicate that at high pressures compact sands become still more compact during shear (i.e., dilate with the same sign as the mellow sands), hence it should be considered that ρ_* depends slightly on p (it is not excluded that this is associated with the effect of granulation of the particles). The quantities Λ and φ in the model are given as functions of one argument, hence the dependence $\varphi(\Lambda)$ can be sought. Indeed, the angle φ is smaller in mellow sands, and greater in compact sands [6, 9, 12, 13].

3. Plastic potential. Now, let us find a potential function $\Psi(\sigma_{ij})$ such that

$$de_{ij} = (\partial\Psi / \partial\sigma_{ij})\, d\lambda \tag{3.1}$$

It can be shown that the governing relations (2.4) correspond to the following form of the potential surfaces

$$\Psi(\sigma_{ij}) = (T^2 - (\Lambda\sin\varphi)\, p^2 - 2c\,(\Lambda\cos\varphi)\, p) = \text{const} \tag{3.2}$$

Let us represent the potential surfaces in the stress space pT and let us determine the constant in the equipotential equation from the condition of its intersection with the Mohr-Coulomb line at the point (T_0, p_0). We obtain

$$T^2 - (\Lambda\sin\varphi)(p + H)^2 = T_0(p_0 + H)(\sin\varphi - \Lambda) \tag{3.3}$$

Let $\sin\varphi > \Lambda > 0$, then (3.3) turns out to be the equation of a hyperbola, whose asymptote has a slope $\sqrt{\Lambda\sin\varphi}$ to the pressure axis p less than the Mohr-Coulomb line.

In a particular case (if only it is possible) $\Lambda = \sin\varphi$, the hyperbola (3.3) degenerates into a line which hence coincides with the Mohr-Coulomb line. In other words, for $\Lambda = \sin\varphi$ the equipotential Ψ = const itself turns out to be the flow surface Φ_σ = const, i.e. the vector of the strain increments is normal to the flow surface. Therefore, this particular case corresponds to application of the associative law to the flow function, as has beeen recommended in [15]. However, it has been disclosed in tests that the dilatancy velocity is not only much less than the value $\Lambda = \sin\ \varphi$, but can also be a negative quantity. Moreover, for $\Lambda = \sin\ \varphi$ the dissipation W is determined only by cohesion of the medium (see Sect. 6 [15] and also [16]), and for $c = 0$ (an ideal granular medium) it turns out that $W = 0$ also. The unreality of the absence of dissipation in a medium with dry friction is often an argument against the application of the associative law in any form. In fact, the remark made refers to the application of the associative law to the flow surface itself.

If $\Lambda < 0$, then Eq. (3.5) turns out to be the equation of an ellipse. For $\Lambda = 0$ we obtain that the equipotential surface becomes a line parallel to the pressure axis.

The cases considered above, corresponding to mellowing ($\Lambda > 0$) and compaction ($\Lambda < 0$), as well as the case of the critical state of incompressibility $\Lambda = 0$ are presented in Fig. 1, where the variable angle of friction φ has also been taken into account. The plastic states therefore fill the angle aOc or bOc, where displacement of the flow surface from the initial aO (or bO) to the critical cO occurs during the development of strains. The critical surface cO can also be considered a failure surface.

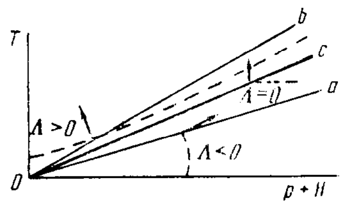

Fig. 1.

Apparently not all states of the medium from the angle aOc can be realized, particularly because of the possible instability of the strain process. This remark refers, firstly, to strain with deconsolidation of the medium for an accompanying reduction in the angle of friction. It is essential that the cylindrical samples of compact sand fail under axial compression, along some narrow zones (the slip surfaces) while the samples of mellow sand fail by taking on barrel-shaped form.

Let us also note that real granular materials are crushed in the high pressure domain to which a break in the Mohr-Coulomb line and a change in the effective angles of friction, cohesion, and dilatancy correspond [17].

The governing relations (2.4) (or in the form (3.1), (3.2))close the system of motion and continuity equations, in combination with the flow condition (2.1) and the hardening laws. The traditional addition of elastic strains result in a comparatively slight change in the strains in the plastic states sector, but the rigid domain adjoining the hydrostatic compression axis is replaced by the elastic strain domain. Correspondingly, it turns out that hydrostatic compression should be elastic.

The fundamental models proposed earlier can provisionally be separated into four kinds. In the first kind are models [18, 19] according to which the volume and shear irreversible strains are developed kinematically independently, where the volume strain (density) is a function of the pressure (the loading and unloading branches are distinct). The governing relations for the shear strains are given either as in deformation plasticity theory [18], or from the condition of similarity of the tensors, the stress deviators and the strain rates [19]. This latter modification corresponds to applying the associative law to equipotentials parallel to the p axis in the pT plane, i.e., which are not coincident with the Mohr-Coulomb line.

It is seen from test results that under purely hydrostatic compression samples of granular medium remain primarily elastic down to pressures at which mass crushing of the grains (sand) occurs, [20], i.e., the irreversible part of the strain is one-third or one-half of its elastic part under primary loading. This deduction was confirmed by tests of L.S. Kozachenko in the IFZ Akad. Nauk SSSR. If the compression is performed for unequal principal stresses, the irreversible changes in volume become quite large [20-22], where the relation between ρ and p turns out to be ambiguous even during loading. For example, it depends on the ratio between the pricipal stresses, and diminution in the consolidation occurs for high ratios even as p grows [22]. On the other hand, at very high hydrostatic compressions when crushing is quite substantial [14], the preeminent dependence of the irreversible volume strains on the pressure itself is indisputable.

In the second kind are models utilizing application of the associative law to the flow function [15, 23]. The medium turns out to be rigid (or elastic) under hydrostatic compression, and to become mellow without limit $\Lambda = \sin \varphi$ under shear. To take account of the consolidation of the medium observable in tests, as well as the hardening effects, generalized models [24 - 27] have been proposed which closed flow surfaces (in the space of principal stresses) have been introduced.

Models of the second kind take account of dilatancy effects, but for states corresponding to the lateral part of the flow surface (the line $T = p \sin \varphi$ on the pT plane), these models predict only mellowing, where it is too intensive (in tests $\Lambda < \sin \varphi$). In this connection, let us also note [28] where it is asserted that a direct experiment does not verify the orthogonality of de_{ij} to $\Phi_\sigma = 0$ for soils. Furthermore, sand remains elastic down to achievement of the Coulomb line in the L.S. Kozachenko tests (the loading path is along the p axis to the point p_A, then along the line $p_A =$ const on the pT plane). This latter means that there cannot be a closed flow surface for which the "bottom" passes to the left of the line $p_A =$ const.

Plastic strain models (the third kind) are also possible, which utilize the conception of the associative flow law, but are based on experimental measurements of the plastic equipotentials. The paper [29] in which strain increments for states of stress corresponding to points under the Mohr-Coulomb line were measured, was a contribution to this aspect. The elastic part was separated out of the total increments, and then the equipotentials $\Psi =$ const were constructed by means of the vector of the plastic strain increments. The shape of these lines was almost elliptical, but they intersected the Mohr-Coulomb line. However, it is essential that the absolute value of the vector de_{ij} at points on the Mohr-Coulomb line be considerably greater than at points below it [29, 30].

Finally, let us note the constructions in [6, 31] (models of the fourth kind), in which it is proposed to consider the irreversible volume strain to be additive: from the part due to the rise in hydrostatic pressure and the part associated with the shear. The associative law applied to the surface $\Phi_\sigma = 0$, was used in [31] for the dilatancy component, and the rate of dilatancy in [6] was considered an independent parameter. It is to be hoped that such models will turn out to be useful in taking account of the effects of both crushing and repacking of the particles of the medium.

It should be emphasised that checking of the homogeneous strain of the samples was not accomplished by far in all the experiments. Hence, further tests (such as are described in [12]) can insert additional corrections in the construction and selection of adequate mathematical models.

4. Problem of deformation of a layer. Let us consider the deformation of a layer of thickness h during which the components of the displacement velocity along the axes 1, 2 are distributed as follows:

$$u_1 = \frac{U}{h} x_2, \qquad u_2 = \frac{V}{h} x_2 \qquad 0 \leqslant x_2 \leqslant h \tag{4.1}$$

where U, V are some constants (the maximum velocities along the horizontal and vertical). The dilatancy condition (2.2) results in the dependence

$$V^2 = \frac{\Lambda^2}{1-\Lambda^2} U^2 \tag{4.2}$$

The following homogeneous state of stress corresponds to the velocity distribution (4.1) if $c \equiv 0$

$$\sigma_{11} = -\frac{1}{2\lambda^{\cdot}} \frac{1+\Lambda\sin\varphi}{\sin\varphi\sqrt{1-\Lambda^2}} \frac{U}{2h}, \qquad \sigma_{22} = -\frac{1}{2\lambda^{\cdot}} \frac{1-\Lambda\sin\varphi}{\sin\varphi\sqrt{1-\Lambda^2}} \frac{U}{2h}$$
$$\sigma_{12} = \frac{1}{2\lambda^{\cdot}} \frac{U}{2h} \tag{4.3}$$

If the specific stress resultants $P = -\sigma_{22}$ and $Q = \sigma_{12}$ are measured on the boundaries $x_2 = \pm h$ then the following relation between them exists

$$Q = KP, \qquad K = \sqrt{1-\Lambda^2}(1-\Lambda\sin\varphi)^{-1}\sin\varphi \tag{4.4}$$

There is a tendency to produce such a flow in shear instruments [12]. The traditional elementary interpretation of the quantity K as the coefficient of internal friction ($K = \operatorname{tg}\varphi_b$) hence results in a deduction on the variability of φ_b (even for a constant true angle $\varphi = \text{const}$). It is curious that the dependence (4.4) even indicates the existence of some peak value of φ_b (Fig. 2, where $\varphi = 30°$, and the dashes correspond apparently to unreal values of Λ) for the domain $\Lambda > 0$, i.e., for $\rho > \rho_*$, as is characteristic for shear tests in compact sand [12-14]. Therefore, data on the coefficeint (angle) of friction obtained even in the case of the simplest plane, laminar flow cannot be referred directly to the true angle of friction φ which enters in the limit relationship (3.1).

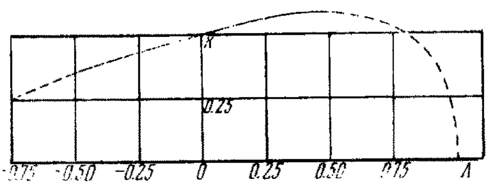

Fig. 2.

When the loading paths $Q(t)$, $P(t)$ are given in order to find the corresponding strains it is necessary first to differentiate condition (4.4), where if $\Lambda = \Lambda(\rho / \rho_*)$ and $\varphi = \varphi(\rho / \rho_*)$, then $K = K(e)$, hence we obtain

$$dQ = KdP = PK_e' de \tag{4.5}$$

On the other hand, the relation

$$\tfrac{1}{4}\sqrt{1-\Lambda^2}\, de = Q\Lambda\, d\lambda \tag{4.6}$$

follows from (4.3) for the increment de in the volume strain. Substitution of the relationship (4.6) into (4.5) permits expression of the increment $d\lambda$ in terms of the loading increments dQ and dP.

Now, the increment in the shear strains

$$d\varepsilon_{12} = {}^1/_2 \sqrt{1-\Lambda^2} (\Lambda K_e')^{-1} (dQ - K\,dP)\,P^{-1} \tag{4.7}$$

can be computed by means of the expressions (4.3), which completes the computation. The condition $dQ - KdP = 0$ corresponds to neutral loading, and the condition $dQ - KdP < 0$ to unloading. In these cases the plastic strain increments equal zero.

Let us note that simultaneous determination of K and Λ in test data permits finding the empirical relation between $\Lambda(\rho)$, $\varphi(\rho)$

5. Hypothetical closure relationships. Although the dependences $\Lambda(\rho)$ and $\varphi(\rho)$ should be sought by processing test results, certain hypothetical models permitting the calculation of these characteristic relations for special kinds of granular media by means of the known values of the angle of friction between the particles, and other microstructure parameters, can also be proposed. Methods of the mechanics of a continuum with microstructure using the concept of vector-directors and slip lines [32] apparently turn out to be most fruitful in this area.

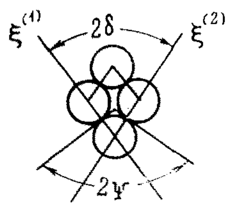

Fig. 3

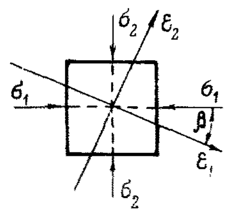

Fig. 4

Let us consider a cell consisting of four touching surfaces (Fig. 3). Idealizing the real picture, we assume these particles to be spherical in shape and of the same radius. The condition that the spacings between the centers of the touching particles remain constant is characteristic for the deformation of the cell (if small elastic strains are excluded from the analysis, see [33]). In other words, the relative displacements of the centers of the spheres should be orthogonal to lines connecting them. If a continuum description of the strain is now introduced, then the mentioned condition indicates the presence of the vector-directors $\xi^{(1)}$, $\xi^{(2)}$ similar to those introduced earlier by Ericksen and Truesdell [34]. The condition of absence of relative displacements of points in the direction of the vector-directors reduces to the following (see (1.3)):

$$\sum_{i,j} \varepsilon_{ij} \xi_i^{(\alpha)} \xi_j^{(\alpha)} = 0, \qquad \alpha = 1,2 \tag{5.1}$$

Condition (5.1) can be transformed [32] to the kinematic interrelation between the volume ε and the shear γ strain rates

$$\varepsilon = -\Lambda \gamma^{\cdot} \operatorname{sgn} \gamma, \quad \Lambda = \cos 2\delta \tag{5.2}$$

where 2δ is the angle between the vectors $\xi^{(1)}$, $\xi^{(2)}$ (Fig. 4). The dependence between the dilatancy rate Λ and the angle δ in (5.2) is easily converted into a dependence on the volume strain (density) of the cell.

The stress resultants on the contacts are connected by the law (1.1) for a relative slippage of the spheres, where a proportionality between the mean pressure and the intensity of the shear stresses acting on the cell corresponds to it [32]. It is essential that the proprtionality coefficient (the effective coefficient of internal friction of the medium as a whole) depend on the volume strain (density) of the medium since the angle δ varies during deformation, as do also the projections of the dry friction forces on the principal axes of the stress tensor [32].

The coefficient of internal friction can even become infinite (the arching or wedging effect), which Orowan [35] noted for spheres. Let us also note that relative displacements of the particles occur along families of lines (slip), mutually orthogonal to the vector-directors which interseet at the angle $2\psi = \pi - 2\delta$.

Consideration of the deformation of an individual cell permits illustration of the condition (2.2) (cf condition (5.2)), as well as the nature of the hardening of a granular medium. However, to obtain quantitative relations it is necessary to examine the statistics of the distribution of such cells by assuming even the presence of their superstructure. Taking account of anisotropy is hence essential.

It is considerably simpler to seek the closure relations by introducing just one velocity field in the neighborhood of the macropoint, but meanwhile utilizing reasoning based on the micropicture of the deformation process.

Such a path is not new in the mechanics of granular media. The most complete papers of this kind are [16, 36, 37] whose authors requirement that the characteristics of the velocity field along which velocity discontinuities are possible, should certainly coincide with lines on which the Coulomb law (1.1) is satisfied. These latter turn out to be characteristics of the stress field if $R = \tau_n$, $N = \sigma_n$ [38], where τ_n, σ_n are the shear and normal components of the stresses on this line (n is the subscript for its normal). Since the particular case of an incompressible medium is examined in [16, 36, 37], the two mentioned families of characteristics may not coincide when the stress tensors and velocities are coaxial. The requirements of coaxiality and the requirement for relative rotation of the principal axes of thestress tensors (so that at least one pair of velocity and stress characteristics would coincide) are hence rejected.

Rejection of coaxiality in the case of an isotropic medium is not acceptable for many reasons. Firstly, the slip lines along which the law (1.1) should be satisfied is a line of strong tangential discontinuity in the velocity, i.e. the shear strain velocity along it becomes infinite, and the concept of the strain rate tensor thereby becomes meaningless at this point. Hence, at points where the ultimate equilibrium conditions of a continuum are satisfied but there are no strong discontinuities, it is not necessary to require coincidence of the lines along which the law (1.1) is satisfied with the velocity field characteristics; it is thereby not necessary to reject the mentioned coaxiality condition.

Secondly, in an isotropic medium the maximum dissipation power of the mechanical work W corresponds to the coaxiality condition. Let us first assume that the principal axes of the mentioned tensor in an element (a square with side $2l$) of the medium (Fig. 4) do not coincide (the principal values are σ_1, σ_2, ε_1, ε_2), respectively). Utilizing the representation (1.3) we find

$$W = 2\sigma_1 \Delta u_1 + 2\sigma_2 \Delta u_2 = (\sigma_1 + \sigma_2)(\varepsilon_1 + \varepsilon_2) l + (\sigma_1 - \sigma_2)(\varepsilon_1 - \varepsilon_2) l \cos 2\beta \quad (5.3)$$

from which it follows that the maximum of W is achieved for $\beta = 0$. Finally, special tests [12] verify that the increments in the plastic strains are coaxial to the stresses.

In passing, let us note that upon compliance with the relations (2.4) the expression (5.3) can be represented as

$$W = (\sigma_1 - \sigma_2)(\varepsilon_1 - \varepsilon_2)(1 - \Lambda / \sin \varphi) + 2c\,\Lambda\,(\varepsilon_1 - \varepsilon_2) \operatorname{ctg} \varphi, \quad l = 1$$

and for $\Lambda = \sin \varphi$, $c = 0$, i.e., for the form [15] of the associative law $W \equiv 0$.

Constructing a hypothetical picture of the strain under conditions of Coulomb frictional force action, let us start from the fact that we find the analog of the slip lines and the analog of the vector-directors in a mean velocity field.

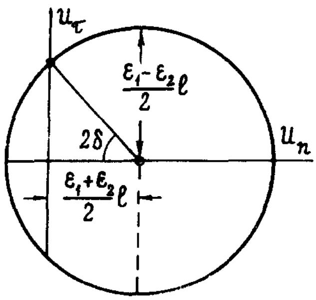

Fig. 5.

The existence of a Mohr circle (Fig. 5) for the tensor ε_{ij} follows from the relationship (1.3) which can also be expressed in terms of the relative velocities

$$u_\tau^2 + (u_n - {}^1/_2(\varepsilon_1 + \varepsilon_2)l)^2 = ({}^1/_2(\varepsilon_1 - \varepsilon_2)l)^2$$

where u_n, u_τ are the velocity components of particles separated by l from the point x_j, along l and perpendicular to it. Because of condition (2.2) which has the form $(\varepsilon_1 + \varepsilon_2) = \Lambda|\varepsilon_1 - \varepsilon_2|$ in the principal strains ε_1, ε_2, the circle (5.4) intersects the axis $u_n = 0$. Hence, two directions exist (forming the angles (*) $\pm\,\delta$; $\Lambda = \cos 2\delta$ with the σ_2, ε_2 axes), on which the spacing between the particles does not change. These directions coincide with the velocity field characteristics; its prototype is the vector-director. The analogs of the directions along which the microslip of the particles on the velocity characteristics seems to occur are the directions u_τ, i.e., the family of the slip lines [9] mutually orthogonal to the velocity field characteristics. Along precisely these does the purely tangential relative particle displacement occur. These slip lines merge with the velocity field characteristics only for $\Lambda \to 0$. The line of strong tangential discontinuity in the velocity is the boundary for domains of continuous plastic strain, and the conditions thereon are boundary conditions.

That state of a continuum when the Coulomb dry friction law (1.1) is realized on the ultimate equilibrium area, but between the force components R, N directed at the angle β and $\beta + \pi/2$, respectively, has been examined earlier [39, 40]. In this case the effective angle of friction is $\varphi = \vartheta + \beta$.

Now, if it is assumed that the direction of the force R is collinear with the slip line, we then obtain [9] a relation between the instantaneous angle of friction φ of the continuum, the dilatancy velocity Λ and the angle of friction ϑ between the particles

$$\varphi = 2\vartheta + \arcsin \Lambda \tag{5.5}$$

This relationship approximates well the relation suggested in [13] on the basis of energy considerations. To find the relation between the dilatancy velocity Λ and the density ρ of the medium, let us first differentiate the expression $\Lambda = \cos 2\delta$ with respect to time. We then find the rate of change of the angle

$$\frac{d\delta}{dt} = -\frac{1}{2}\frac{1}{\sqrt{1-\Lambda^2}}\frac{d\Lambda}{dt} \tag{5.6}$$

Now the hypothesis that the relative velocity of particles on the characteristic agrees with the velocity of displacement of the characteristic itself (this condition is satisfied strongly for the true vector-directors frozen in the medium; on the other hand, the

(*) The quantity $\nu = \pi/2 - 2\delta$ is called the dilatancy angle [12].

assumption of their presence contradicts the condition of invariant isotropy of the medium)

$$u_\tau(\delta) = {}^1/_2\,(\varepsilon_1 - \varepsilon_2)\, l \sin 2\delta = l\,(d\delta\,/\,dt) \tag{5.7}$$

results in the relationship

$$(\varepsilon_1 - \varepsilon_2)\sqrt{1 - \Lambda^2} = \left(\sqrt{1 - \Lambda^2}\right)^{-1} d\Lambda\,/\,dt \tag{5.8}$$

This latter can be represented as

$$\varepsilon dt = -\rho^{-1} d\rho = -(1 - \Lambda^2)\,\Lambda\, d\Lambda \tag{5.9}$$

Integrating (5.9) results in the relations (Fig. 6, where curves of the change in sample density under shear are presented)

$$\begin{aligned} \Lambda &= -\sqrt{1 - (\rho\,/\,\rho_*)^2}, \qquad d\Lambda > 0 \quad \text{for } \Lambda < 0 \\ \Lambda &= \sqrt{1 - (\rho_*\,/\,\rho)^2}, \qquad d\Lambda < 0 \quad \text{for } \Lambda > 0 \end{aligned} \tag{5.10}$$

The hypothetical relationships (5.5) and (5.9) close the rigidly plastic hardening model of a granular medium presented above. They are useful as approximations although they certainly cannot replace the functions $\Lambda(\rho)$, $\varphi(\rho)$ for real materials. In particular, it is

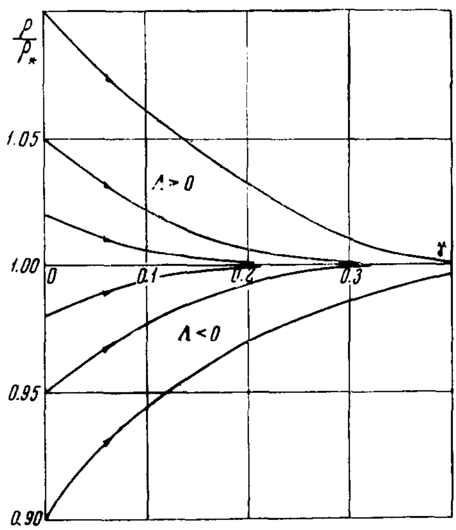

Fig. 6.

essential that the derivative $d\varphi\,/\,d\Lambda = (1 - \Lambda^2)^{-1/2}$ does not vanish as $\Lambda \to 0$ but this does not assure a smooth passage to the limit of the hardening plastic model to the model of an incompressible ($\Lambda = 0$) flow without hardening, which corresponds to the critical state (failure) of a medium.

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BIBLIOGRAPHY

1. Sedov, L. I., Continuum Mechanics. Vol. 1, 2, Moscow, "Nauka", 1970.
2. Olszak, W., Mroz, Z. and Perzyna, P., Recent Trends in the Development of the Theory of Plasticity. Pergamon Press, 1963.

3. Nikolaevskii, V. N., On the theory of inelastic strains of porous media. In the collection: Theory and Practice of Oil Production (pp. 75-87) Moscow, "NEDRA", 1966.
4. Nikolaevskii, V. N., On the thermodynamic analysis of models of an inelastic continuum. Inzh. Zh. Mekh. Tverd. Tela, №3, 1966.
5. Nikolaevskii, V. N., Monochromatic waves in an elastic medium with the local appearance of dry friction. Inzh. Zh. Mekh. Tverd. Tela, №4, 1968.
6. Nikolaevskii, V. N., On the relations between the volume and shear plastic strains, and on shocks in soft soils. Dokl. Akad. Nauk SSSR, Vol. 177, №3, 1967.
7. Reynolds, O., On the dilatancy of media composed of rigid particles in contact. With experimental illustrations. Phil. Mag., Ser. 5, Vol. 20, №127, 1885.
8. Mises, R. V., Mechanik der plastischen Formänderung von Kristallen. ZAMM Vol. 8, №3, 1928.
9. Nikolaevskii, V. N. and Syrnikov, N. M., On the plane limit flow of dilating granular media. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, №2, 1970.
10. Sawczuk, A. and Stutz, P., On formulation of stress-strain relations for soils at failure. ZAMP, Vol. 19, №5, 1968.
11. Cameron, I. G. and Scorgie, G. C., Dynamics of intense underground explosions. J. Inst. Math. Applic., Vol. 4, №2, 1968.
12. Roscoe, K. H., The influence of strains in soil mechanics. Geotechnique, Vol. 20, №2, 1970.
13. Rowe, P. W., The stress-dilatancy relation for static equilibrium of an assembly of particles in contact. Proc. Roy. Soc. A., Vol. 269, №1339, 1962.
14. Vesic, A. S. and Clough, G. W., Behavior of granular materials under high stress. J. Soil Mech. and Foundat. Div., Proc. Amer. Soc. Civil Engrs. Vol. 94, №3, 1968.
15. Drucker, D. C. and Prager, W., Soil mechanics and plastic analysis or limit design. Quart. Appl. Math., Vol. 10, №2, 1952.
16. Mandl, G. and Fernandez-Lique, R., Fully developed plastic shear flow of granular materials. Geotechnique, Vol. 20, №3, 1970.
17. Hall, C. D. and Harrisberger, W. H., Stability of sand arches: a key to sand control. J. Petrol. Technol., Vol. 22, 1970.
18. Rakhmatulin, Kh. A., Sagomonian, A. Ia., and Alekseev, N. A., Questions of Soil Dynamics. Moscow, MOSCOW UNIVERSITY PRESS, 1964.
19. Grigorian, S. S., On basic concepts in soil dynamics. PMM Vol. 24, №6, 1960.
20. Ko, H. Y. and Scott, R. F., Deformation of sand in hydrostatic compression. J. Soil Mech. and Foundat. Div., Proc. Amer. Soc. Civil Engrs., Vol. 93, №3, 1967.
21. Stoll, R. D. and Ebeido, I. A., Unloading effects in the dynamics response of granular soil. Exptl. Mech., Vol. 6, №10, 1966.
22. El-Sohby, M. A., Deformation of sands under constant stress ratios. Proc. 7-th Intern. Conf. Soil Mech., Found. Eng., Mexico, 1969. Mexico, Vol. I, 1969.
23. Shield, R. T., On Coulomb's law of failure in soils. J. Mech. and Phys. Solids, Vol. 4, №1, 1956.
24. Drucker, D. C., Gibson, R. E. and Henkel, D. J., Soil mechanics and work-hardening theories of plasticity. Trans. Amer. Soc. Civil Engrs., Vol. 122 1957.

25. Jenike, A. W. and Shield, R. T., On the plastic flow of Coulomb solids beyond original failure. J. Appl. Mech., Vol. 26, №4, 1959.
26. Suh, N. P., A yield criterion for plastic, frictional work-hardening granular material. Intern. J. Powder Metallurgy, Vol. 5, №1, 1969.
27. Didukh, B. I. and Ioselevich, V. A., On the construction of a theory of plastic soil hardening. Izv. Akad Nauk SSSR, Mekh. Tverd. Tela, №2, 1970.
28. Haythornthwaite, R. M., Stress and strain in soils. In: Plasticity, Oxford, Pergamon Press, 1960.
29. Porooshasb, H. B., Holubec, I. and Sherbourne, A. N., Yielding and flow of sand in triaxial compression. Canad. Geotechn. J., pt. 1, Vol. 3, №4; Vol. 4, №4, 1967.
30. Dreszer, A. and Bojanowski, W., On the influence of stress path upon the mechanical properties of granular material. Arch. Inz. Ladowej, Vol. 14, №3, 1968.
31. Ivlev, D. D. and Martynova, T. N., On the theory of compressible ideally plastic media. PMM, Vol. 27, №3, 1963.
32. Nikolaevskii, V. N. and Afanasiev, E. F., On same examples of media with microstructure of continuous particles. Intern. J. Solids and Structures, Vol. 5, №7, Pergamon Press, 1969.
33. Deresiewicz, H., Mechanics of granular matter. In: Advances in Appl. Mech., Vol. 5, N. Y., Academic Press, Inc., 1958.
34. Ericksen, G. L. and Truesdell, C., Exact theory of stress and strain in rods and shells. Archs. Ration. Mech. Analysis, Vol. 1, №4, 1958.
35. Orowan, E., Dilatancy and the seismic focal mechanism. Revs. Geophys. Vol. 4, №3, 1966.
36. Geniev, G. A., Problems of the Dynamics of Granular Media. Nauchn. Soobshch. TsNIISK, Akad. Stroit. i Arkhitektury, №2, 1958.
37. Josselin De Jong, G., Statics and Kinematics in the Failable Zone of a Granular Material. Delft, Uitgeverig, Waltman, 1959.
38. Sokolovskii, V. V., Statics of a Granular Medium. Moscow, FIZMATGIZ, 1960.
39. Nikolaevskii, V. N., On the formulation of the defining equations for a plane flow of a continuous medium with dry friction. PMM Vol. 32, №5, 1968.
40. Nikolaevskii, V. N., On a generalization of the Coulomb limit condition for ideally granular continua. Prikl. Mekh., Vol. 5, №3, 1969.

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